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Wigner's little group, gauge transformations and dimensional descent

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Abstract

We propose a technique called dimensional descent to show that Wigner's little group for massless particles, which acts as a generator of gauge transformation for usual Maxwell theory, has an identical role even for topologically massive gauge theories. The examples of $B \wedge F$ theory and Maxwell–Chern–Simons theory are analysed in detail.

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Wigner's little group [1], which is a subgroup of the Poincaré group that leaves the four-momentum invariant, is of fundamental importance for both massive and massless particles. In the latter case this group is $E(2)$, which is a semi-direct product of $T(2)$ (the group of translations in the plane) and $SO(2)$. Each of these subgroups has its own role, but here we are concerned with $T(2)$ which acts as a generator of gauge transformation [2, 3] in usual gauge theories, such as Maxwell or Kalb–Ramond [4], where the quanta of excitations are massless. There are, however, instances where gauge invariance coexists with mass—the topologically massive gauge theories such as the $B \wedge F$ [5] theory or the Maxwell–Chern–Simons (MCS) [6] theory in 2 + 1 dimensions. Recently we showed [7] that in the $B \wedge F$ theory, the gauge invariance is generated by a particular representation of $T(3) \subset E(3)$, so that one has to go beyond Wigner's little group. For the MCS model, however, Wigner's little group (isomorphic to $\mathcal{R} \times \mathcal{Z}_2$) itself suffices, albeit in a different representation. Is there then a systematic method of obtaining these distinct representations and, if possible, connect them to Wigner's little group for massless particles, which acts as gauge generator for conventional gauge theories?

The object of this paper is to analyse this and related issues. We show that the specific representations of the translation groups that act as gauge generators of topologically massive gauge theories follow naturally by a dimensional descent of the standard representation of Wigner's little group in one higher dimension. As a byproduct we show the connection between the helicity quantum numbers of massless particles and the spin quantum number of massive particles in one lower dimension.

Let us first recapitulate certain results [7] for the $B \wedge F$ theory in 3 + 1 dimensions, whose dynamics is governed by the following Lagrangian density[1] [5]:

$$\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \frac{1}{12}H_{\mu\nu\lambda}H^{\mu\nu\lambda} - \frac{m}{6}\epsilon_{\mu\nu\lambda\rho}H^{\mu\nu\lambda}A^{\rho} \tag{1}$$

where

$$F_{\mu\nu} = \partial_{[\mu}A_{\nu]} \qquad H_{\mu\nu\lambda} = \partial_{[\mu}B_{\nu\lambda]} \qquad \mu = 0, 1, 2, 3. \tag{2}$$

The polarization vector (tensor) associated with the one (two) form fields A_μ ($B_{\nu\lambda}$) are given by [7]

$$\varepsilon^{\mu} = -\mathrm{i}\begin{pmatrix} 0 \\ a_1 \\ a_2 \\ a_3 \end{pmatrix} \qquad \varepsilon = \{\varepsilon^{\nu\lambda}\} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & -a_2 \\ 0 & -a_3 & 0 & a_1 \\ 0 & a_2 & -a_1 & 0 \end{pmatrix} \tag{3}$$

corresponding to the momentum 4-vector

$$p^{\mu} = (m, 0, 0, 0)^T \tag{4}$$

for a massive quanta at rest. Note that the entries in (3) satisfy a duality relation connecting the space components of the two matrices.

It might be mentioned that identical expressions for the polarization vector (tensor) are obtained for Proca (massive Kalb–Ramond) theory.[2] This is not unexpected if one recalls the equivalence of the $B \wedge F$ model to such theories [8].

To begin with the present analysis, note that $T(3)$, which acts as the generator of gauge transformation in the $B \wedge F$ model [7], is an Abelian invariant subgroup of $E(3)$. The group $E(3)$ is Wigner's little group for a massless particle in five dimensions. This suggests that we shall have to consider 5 (= 4 + 1)-dimensional space–time. Now an element of Wigner's little group in five dimensions can be written as

$$W_5(p, q, r; \psi, \phi, \eta) = \begin{pmatrix} 1 + \frac{p^2+q^2+r^2}{2} & p & q & r & -\frac{p^2+q^2+r^2}{2} \\ p & & & & -p \\ q & & R(\psi, \phi, \eta) & & -q \\ r & & & & -r \\ \frac{p^2+q^2+r^2}{2} & p & q & r & 1 - \frac{p^2+q^2+r^2}{2} \end{pmatrix} \tag{5}$$

where p, q, r are any real numbers, while $R(\psi, \phi, \eta) \in SO(3)$, with (ψ, ϕ, η) being a triplet of Euler angles. The above result can be derived by following the standard treatment [9]. The corresponding element of the translational group $T(3)$ can be trivially obtained by setting $R(\psi, \phi, \eta)$ to be the identity matrix and will be denoted by $W_5(p, q, r) = W_5(p, q, r; \mathbf{0})$.

Let us now consider free Maxwell theory in five dimensions,

$$\mathcal{L} = -\tfrac{1}{4}F^{ij}F_{ij} \qquad i, j = 0, 1, 2, 3, 4. \tag{6}$$

A free photon will have three independent transverse degrees of freedom. Correspondingly the polarization vector ε^i can be brought to the following form:

$$\varepsilon^i = (0, a_1, a_2, a_3, 0)^T \tag{7}$$

[1] Greek indices will always denote (3 + 1)-dimensional space–time. Latin indices from the middle of the alphabet (such as i, j, k) will indicate 4 + 1 dimensions while those from the beginning (such as a, b, c) will correspond to 2 + 1 dimensions.

[2] Of course, the entries in the different matrices are now independent and are not constrained by the duality relation mentioned earlier which is a consequence of the coupling term in the $B \wedge F$ model.

if the photon of energy 'ω' is taken to be propagating in the 4-direction so that the energy momentum 5-vector takes the following form:

$$p^i = (\omega, 0, 0, 0, \omega)^T. \tag{8}$$

Note that this automatically satisfies the 'Lorentz gauge' $\varepsilon^i p_i = 0$. If we now suppress the last rows of the column matrices ε^i (7) and p^i (8) we end up with the polarization vector and energy–momentum 4-vector of a Proca quanta of mass $m = \omega$ in the rest frame in 3 + 1 dimensions. This is equivalent to applying the projection operator given by the matrix

$$\mathcal{P} = \mathrm{diag}(1, 1, 1, 1, 0) \tag{9}$$

and deleting the last rows which now have only null entries. This method of getting results in any dimension by starting from the known results in one dimension higher will be called dimensional descent.

Similarly, to reproduce the polarization tensor of massive Kalb–Ramond (KR) theory in 3 + 1 dimensions let us consider the free massless KR model in five dimensions,

$$\mathcal{L} = \tfrac{1}{12} H^{ijk} H_{ijk}. \tag{10}$$

Proceeding just as in [7], the polarization matrix $\varepsilon = \{\varepsilon^{ij}\}$ can be brought to the following maximally reduced form:

$$\varepsilon = \{\varepsilon^{ij}\} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \varepsilon^{12} & \varepsilon^{13} & 0 \\ 0 & -\varepsilon^{12} & 0 & \varepsilon^{23} & 0 \\ 0 & -\varepsilon^{13} & -\varepsilon^{23} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{11}$$

Again deleting the last row and column, one gets the polarization tensor $(\bar{\varepsilon})$ (3) of the massive KR model. This is equivalent to applying the projection operator as $\mathcal{P}\varepsilon\mathcal{P}$. Thus the polarization vector and tensor of the Proca and massive KR models, respectively, have been reproduced.

Now coming to the gauge transformation properties of the polarization vector (7) and polarization tensor (11) under the translational subgroup $T(3)$, let $W_5(p, q, r)$ act on these objects one by one. First, acting on ε^i(7), one gets

$$\varepsilon^i \to \varepsilon'^i = W_5(p, q, r)^i{}_j \varepsilon^j = \varepsilon^i + \delta\varepsilon^i = \varepsilon^i + (pa_1 + qa_2 + ra_3)\frac{p^i}{\omega} \tag{12}$$

which is indeed a gauge transformation in (4 + 1)-dimensional Maxwell theory. Applying the projection operator $\mathcal{P}$ (9) on (12) yields

$$\delta\bar{\varepsilon}^\mu = \mathcal{P}\delta\varepsilon^i = \frac{1}{\omega}(pa_1 + qa_2 + ra_3)p^\mu. \tag{13}$$

Here $\bar{\varepsilon}^\mu = (0, a_1, a_2, a_3)^T$ and, modulus the i-factor, corresponds to the expression in (3). This is precisely how the polarization vector in $B \wedge F$ theory transforms under gauge transformation [7]. In fact we can write

$$\delta\bar{\varepsilon}^\mu = D^\mu{}_\nu(p, q, r)\bar{\varepsilon}^\nu - \bar{\varepsilon}^\mu \tag{14}$$

where

$$D(p, q, r) = \begin{pmatrix} 1 & p & q & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{15}$$

Coming next to the polarization matrix $\varepsilon = \{\varepsilon^{ij}\}$, its transformation law is given by

$$\varepsilon \to \varepsilon' = W_5(p, q, r)\varepsilon W_5^T(p, q, r) = \varepsilon + \delta\varepsilon$$

where

$$\delta\varepsilon = \{\delta\varepsilon^{ij}\} = \begin{pmatrix} 0 & \alpha_1 & \alpha_2 & \alpha_3 & 0 \\ -\alpha_1 & 0 & 0 & 0 & -\alpha_1 \\ -\alpha_2 & 0 & 0 & 0 & -\alpha_2 \\ -\alpha_3 & 0 & 0 & 0 & -\alpha_3 \\ 0 & \alpha_1 & \alpha_2 & \alpha_3 & 0 \end{pmatrix} \tag{16}$$

where $\alpha_1 = -(q\varepsilon^{12} + r\varepsilon^{13})$, $\alpha_2 = (p\varepsilon^{12} - r\varepsilon^{23}))$, $\alpha_3 = (p\varepsilon^{13} + q\varepsilon^{23})$. Again this can be easily recognized as a gauge transformation in (4 + 1)-dimensional KR theory involving massless quanta, as $\delta\varepsilon^{ij}$ can be expressed as $\approx(p^i f^j(p) - p^j f^i(p))$ with a suitable choice for $f^i(p)$, where p^i is of the form (8). Now applying the projection operator $\mathcal{P}$ (9), we get the change in the (3 + 1)-dimensional polarization matrix $\bar{\varepsilon} = \{\varepsilon^{\mu\nu}\}$, by the formula $\delta\bar{\varepsilon} = \mathcal{P}\delta\varepsilon\mathcal{P}^T$. This simply amounts to a deletion of the last row and column of $\delta\varepsilon$. The result can be expressed more compactly as

$$\delta\bar{\varepsilon} = (D\bar{\varepsilon}D^T - \bar{\varepsilon}) \tag{17}$$

where D has already been defined in (15). Again this has the precise form of gauge transformation of the polarization matrix of the $B \wedge F$ model [7], since it can be cast in the form

$$\delta\bar{\varepsilon}_{\mu\nu} = \mathrm{i}(p_\mu f_\nu(p) - p_\nu f_\mu(p)) \tag{18}$$

for a suitable $f_\mu(p)$, where the form of p_μ is now given by (4).

Clearly the generators $T_1 = \frac{\partial D}{\partial p}$, $T_2 = \frac{\partial D}{\partial q}$, $T_3 = \frac{\partial D}{\partial r}$ provide a Lie algebra basis for the Abelian group $T(3)$, as they commute mutually. Note that they also satisfy

$$(T_1)^2 = (T_2)^2 = (T_3)^2 = T_1T_2 = T_1T_3 = T_2T_3 = 0 \tag{19}$$

so that

$$D(p, q, r) = \mathrm{e}^{pT_1 + qT_2 + rT_3} = 1 + pT_1 + qT_2 + rT_3. \tag{20}$$

The change in the polarization vector $\bar{\varepsilon}$ (14) can be expressed as the action of a Lie algebra element

$$\delta\bar{\varepsilon}^\mu = (pT_1 + qT_2 + rT_3)\bar{\varepsilon}^\mu. \tag{21}$$

Besides this D also preserves the 4-momentum of a particle at rest. This representation of $D(p, q, r)$ was earlier introduced in [7]. However, here we have shown it can be connected to the Wigner's little group for a massless particle in 4 + 1 dimensions through appropriate projection in the intermediate steps, where the massless particles moving in 4 + 1 dimensions can be associated with a massive particle at rest in 3 + 1 dimensions. Similarly, the polarization vector and tensor of $B \wedge F$ theory in 3 + 1 dimensions can be associated with the polarization vector and polarization tensor of the free Maxwell and KR theories in 4 + 1 dimensions.

At this stage, one can ask whether the same features of dimensional descent from higher 3 + 1 dimensions will go through for the topologically massive Maxwell–Chern–Simons theory in 2 + 1 dimensions. For that let us start with free Maxwell theory in 3 + 1 dimensions. This has two transverse degrees of freedom. Correspondingly, the polarization vector ε^μ takes the following maximally reduced form $\varepsilon^\mu = (0, a, b, 0)^T$, if the 4-momentum p^μ of the photon of energy 'ω' and propagating in the 3-direction takes the following form $p^\mu = (\omega, 0, 0, \omega)^T$. The generator of gauge transformation in this case is $T(2)$, which is a subgroup of $E(2)$. Now the representation of the (3 + 1)-dimensional Wigner's little group $W_4(p, q, \phi)$ is just obtained from $W_5(p, q, r; \psi, \phi, \eta)$ by replacing $R(\psi, \phi, \eta)$ by $R(\phi) \in SO(2)$, by deleting the columns and rows involving r and finally setting $r = 0$ elsewhere. To obtain a (4×4) representation of $T(2)$ one just sets $\phi = 0$ in $W_4(p, q, \phi)$.

We are now ready to discuss the dimensional descent from 3 + 1 to 2 + 1 dimensions. As a first step, this involves reducing the last rows of ε^μ and p^μ to zero by using the projection operator $\mathcal{P} = \mathrm{diag}(1, 1, 1, 0)$. The descended objects $\bar{\varepsilon}^a = (0, a_1, a_2)^T$ $(a = 0, 1, 2)$ and $p^a = (\omega, 0, 0)^T$ correspond to the polarization vector and momentum 3-vector of a Proca quanta of mass ω at rest, of the corresponding Proca theory

$$\mathcal{L} = -\frac{1}{4}F^{ab}F_{ab} + \frac{\omega^2}{2}A^a A_a \tag{22}$$

in 2 + 1 dimensions. In order to discuss the gauge transformation properties it is essential to provide a 3×3 representation of $T(2)$ (denoted by $\bar{D}(p, q)$) which amounts to deleting the last row and column of $D(p, q, r)$ in (15)

$$\bar{D}(p, q) = \begin{pmatrix} 1 & p & q \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{23}$$

The corresponding generators are given by

$$\bar{T}_1 = \frac{\partial \bar{D}}{\partial p} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad \bar{T}_2 = \frac{\partial \bar{D}}{\partial q} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \tag{24}$$

Just as the Proca theory in 3 + 1 dimensions maps to the $B \wedge F$ theory, where gauge transformations were discussed, the Proca theory in 2 + 1 dimensions is actually a doublet of Maxwell–Chern–Simons theories [10, 11]:

$$\mathcal{L} = \mathcal{L}_+ \oplus \mathcal{L}_- \tag{25}$$

where

$$\mathcal{L}_\pm = -\frac{1}{4}F^{ab}F_{ab} \pm \frac{\theta}{2}\epsilon^{abc}A_a\partial_b A_c \tag{26}$$

with each of $\mathcal{L}_+$ or $\mathcal{L}_-$ being a topologically massive gauge theory. The mass of the MCS quanta is $\omega = |\theta|$, where ω is the parameter entering in (22). We can therefore study the gauge transformation generated in this doublet. The polarization vector for $\mathcal{L}_\pm$, with only one degree of freedom for each of $\mathcal{L}_+$ and $\mathcal{L}_-$, has been found to be [12, 13]

$$\bar{\varepsilon}^a_\pm = \frac{1}{\sqrt{2}}(0, 1, \pm \mathrm{i})^T \tag{27}$$

while the 3-momentum p^a obviously takes the same form as in the Proca model. In analogy with (21), here also one can write

$$\delta\bar{\varepsilon}^a = (p\bar{T}_1 + q\bar{T}_2)^a{}_b\bar{\varepsilon}^b \tag{28}$$

where $\bar{\varepsilon}^a = (0, a_1, a_2)^T$ is the polarization vector for the Proca theory in 2 + 1 dimensions. Had the Proca theory been a gauge theory, (28) would have represented a gauge transformation, as it can be written as

$$\delta\bar{\varepsilon}^a = \frac{pa_1 + qa_2}{\omega}p^a. \tag{29}$$

But since Proca theory is not a gauge theory, we can only study the gauge transformation properties of each of the doublet $\mathcal{L}_\pm$ (26) individually. First note that the Proca polarization vector $\bar{\varepsilon}^a$ is just a linear combination of the two real orthonormal canonical vectors ε_1 and ε_2 where

$$\bar{\varepsilon}^a = a_1\varepsilon_1 + a_2\varepsilon_2 \qquad \varepsilon_1 = (0, 1, 0)^T \qquad \varepsilon_2 = (0, 0, 1)^T. \tag{30}$$

Correspondingly, the generators $\bar{T}_1$ and $\bar{T}_2$ (24) form an orthonormal basis as they satisfy $\mathrm{tr}(\bar{T}_a^\dagger \bar{T}_b) = \delta_{ab}$. Furthermore,

$$\bar{T}_1\varepsilon_1 = \bar{T}_2\varepsilon_2 = (1, 0, 0)^T = \frac{p^a}{\omega} \qquad \bar{T}_1\varepsilon_2 = \bar{T}_2\varepsilon_1 = 0. \tag{31}$$

On the other hand, the polarization vectors $\bar{\varepsilon}^a_+$ and $\bar{\varepsilon}^a_-$ (27) also provide an orthonormal basis (complex) in the plane as

$$(\bar{\varepsilon}^a_+)^\dagger(\bar{\varepsilon}^a_-) = 0 \qquad (\bar{\varepsilon}^a_+)^\dagger(\bar{\varepsilon}^a_+) = (\bar{\varepsilon}^a_-)^\dagger(\bar{\varepsilon}^a_-) = 1 \tag{32}$$

and can be obtained from the above mentioned canonical ones by appropriate $SU(2)$ transformation. This suggests that we consider the following orthonormal basis for the Lie algebra of $T(2)$:

$$\bar{T}_\pm = \frac{1}{\sqrt{2}}(\bar{T}_1 \mp \mathrm{i}\bar{T}_2) = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & \mp\mathrm{i} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{33}$$

instead of $\bar{T}_1$ and $\bar{T}_2$. Note that they also satisfy relations similar to the $\bar{T}_1$–$\bar{T}_2$ basis,

$$\mathrm{tr}(\bar{T}_+^\dagger \bar{T}_+) = \mathrm{tr}(\bar{T}_-^\dagger \bar{T}_-) = 1 \qquad \mathrm{tr}(\bar{T}_+^\dagger \bar{T}_-) = 0. \tag{34}$$

One can now easily see that

$$\bar{T}_+\varepsilon_+ = \bar{T}_-\varepsilon_- = \frac{p^a}{\omega} \qquad \bar{T}_+\varepsilon_- = \bar{T}_-\varepsilon_+ = 0 \tag{35}$$

analogous to (31). Furthermore,

$$\delta\bar{\varepsilon}^a_\pm = \alpha_\pm \bar{T}_\pm \bar{\varepsilon}^a_\pm = \frac{\alpha_\pm}{\omega} p^a. \tag{36}$$

This indicates that $\bar{T}_\pm$—the generators of the Lie algebra of $T(2)$ in the rotated (complex) basis—generate independent gauge transformations in $\mathcal{L}_\pm$ respectively. One therefore can understand how the appropriate representation of the generator of gauge transformation in the doublet of MCS theory can be obtained from the higher (3 + 1)-dimensional Wigner group through dimensional descent. A finite gauge transformation is obtained by integrating (36), i.e. exponentiating the corresponding Lie algebra element. This gives two representations of Wigner's little group for massless particles in 2 + 1 dimensions, which is isomorphic to $\mathcal{R} \times \mathcal{Z}_2$, although here we are just considering the component which is connected to the identity

$$G_\pm(\alpha_\pm) = \mathrm{e}^{\alpha_\pm \bar{T}_\pm} = 1 + \alpha_\pm \bar{T}_\pm = \begin{pmatrix} 1 & \frac{\alpha_\pm}{\sqrt{2}} & \mp\mathrm{i}\frac{\alpha_\pm}{\sqrt{2}} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{37}$$

Note that $G_\pm(\alpha_\pm)$ generate gauge transformation in the doublet $\mathcal{L}_\pm$, and are related by complex conjugation. This complex conjugation is also a symmetry of the doublet. Such representations were earlier considered in [13].

Another aspect of this doublet structure is illuminated by observing that, in the (3 + 1)-dimensional case, the photon has two helicity states (±1), as follows from topological considerations [9]. By our dimensional descent this corresponds to the two spin states of the Proca quanta in 2 + 1 dimensions. Each of these states is shared amongst the two components of the doublet (25). This is consistent with the fact that an explicit computation of the spin in the MCS theory ($\mathcal{L}_{\pm}$) (25) leads to the same values [6].

We conclude by observing the unified picture that emerges from this analysis. It might be recalled that Wigner's little group for massive and massless particles can be regarded as having a common origin. This is because the $E(2)$-like little group for massless particles is obtained from the $SO(3)$-like little group for massive particles in the infinite momentum/zero mass limit [14]. It is essentially a consequence of the group contraction method connecting these two groups [15]. In this paper we showed the common origin of the gauge generators for usual (i.e., massless) or topologically massive gauge theories. The representations in the latter are obtained by a dimensional descent from one higher dimension of the representation in the former. Following the same ideas it was possible to interpret the spin quantum number of massive particles as the helicity quantum number of massless particles in one higher dimension. As a bonus the structure of the Proca model (in 2 + 1 dimensions) as a doublet of Maxwell–Chern–Simons theories becomes manifest. This is distinctive from the Proca model in 3 + 1 dimensions which gets mapped to the $B \wedge F$ model and is devoid of any doublet structure. Finally, the fact that the gauge generators annihilated the physical states was very similar to the Dirac Hamiltonian formulation [16] where the Gauss operators, which act as gauge generators, share the same property. It would be worthwhile to pursue this connection between Wigner's little group and the Gauss operators.

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